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A comparison between the Sinc–Galerkin and the modified decomposition methods for solving two-point boundary-value problems

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Abstract

One of the new techniques used in solving boundary-value problems involving ordinary differential equations is the Sinc–Galerkin method. This method has been shown to be a powerful numerical tool for finding fast and accurate solutions. A less known technique that has been around for almost two decades is the decomposition method. In this paper we solve boundary-value problems of higher order using these two methods and then compare the results. It is shown that the Sinc–Galerkin method in many instances gives better results.

MSC: 65N30; 35K05

Keywords: Sinc–Galerkin method; The modified decomposition method; Higher-order; Nonlinear

1. Introduction

Recent studies in hydrodynamic and hydromagnetic stability have discovered the existence of a class of characteristic-value problems in differential equations of high order which have genuine mathematical interest.

Experience in solving high-order boundary value problems has shown that insight may be obtained by solving the special problem first of all.

The numerical analysis literature on methods for solving higher-order boundary value problems is comparatively sparse. The book by Agarwal [5] contains theorems on the conditions for existence and uniqueness of the solution, though no numerical methods are given therein. More recently, sixth order and eighth-order boundary value problems are solved in [8,13,24,26].

Accurate and fast numerical solution of two-point boundary value ordinary differential equations is necessary in many important scientific and engineering applications, e.g. boundary layer theory, the study of stellar interiors, control and optimization theory, and flow networks in biology.

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The aim of this paper is to compare the Sinc–Galerkin and the modified decomposition methods in solving linear and nonlinear boundary-value problems of order $2n$, $n = 1, 2, 3$,

$$y^{(2n)} = f(x, y) \tag{1.1}$$

subject to boundary conditions

$$y^{(i)}(0) = A_i, \qquad y^{(i)}(1) = B_i, \quad i = 0, 1, \ldots, n-1. \tag{1.2}$$

where A_i and B_i are finite constants. We assume that y is sufficiently differentiable and that an unique solution of (1.1) exists. It will be shown that although the decomposition method is more popular, the Sinc–Galerkin method gives better results. To the best of our knowledge such a comparison has not been done before.

In recent years, a lot of attention has been devoted to the study of the Sinc–Galerkin method to investigate various scientific models. The Sinc–Galerkin methods for ordinary differential equations has many salient features due to the properties of the basis functions and the manner in which the problem is discretized. Of equal practical significance is the fact that the method's implementation requires no modification in the presence of singularities. The approximating discrete system depends only on parameters of the differential equation regardless of whether it is singular or nonsingular. The error of the method converges to zero like $\mathrm{O}(e^{-k\sqrt{N}})$, as $N \to \infty$, where N is the numerical of collocation points used, and where k is a positive constant independent of N.

The efficiency of the method has been formally proved by many researchers. Yin [31] applied the Sinc-Collocation method to solve singular a problem-like Poisson . Dockery [10] applied the Sinc–Galerkin to handle reaction–diffusion equations, and Bialecki [7] used the Sinc-Collocation methods to solve a two point boundary value problem. Smith et al. [22] applied the Sinc–Galerkin to handle fourth-order ordinary differential equations. El-Gamel et al. [13] used the sinc-Galerkkin method to solve linear sixth-order ordinary differential equation. Also, El-Gamel and Zayed [14] applied the Sinc–Galerkin method to solve nonlinear boundary value problems. Finally, Mohsen and El-Gamel [21] used Sinc-Collocation method for the linear Fredholm integro-differential equations. For more details about Sinc–Galerkin method see [12,15–17,20,23] and the references therein.

The Adomian decomposition method ADM has been applied to wide class of stochastic and deterministic problems in many interesting mathematics and physics areas [1,3,30]. This method has some significant advantages over numerical methods. It provides analytic, verifiable,rapidly convergent approximation which yields insight into the character and the behavior of the solution just as in the closed form solution. Adomian gave a review of the decomposition method in [2].

Several authors have compared the ADM with some existing techniques in solving different types of problems. Bellomo and Monaco [6] have compared the ADM and the perturbation technique when they are used in solving random nonlinear differential equations. Edwards et al. [11] have introduced their comparison of ADM and Runge-Kutta methods for approximate solutions of some predator prey model equations. Wazwaz proposed a new approach to develop a nonperturbative approximate solution for the Thomas–Fermi equation. This approach is based upon a modification of the ADM [27]. Recently, he introduced a comparison between the ADM and the Taylor series method [29]. He showed that the ADM minimizes the computational difficulties of the Taylor series in that the components of the solution are determined elegantly by using simple integrals. Finally, a comparison of Adomians decomposition method and wavelet-Galerkin method for solving integro-differential equations is made by El-Sayed and Abdel-Aziz [18].

The paper is organized as follows. Section 2, we introduce the Sinc–Galerkin method and show how Sinc are used to solve higher-order differential equations numerically. The modified decomposition method is discussed in Section 3. In Section 4, we apply both methods to specific problems, compare the results, and close with conclusions.

2. The Sinc–Galerkin method

The Sinc–Galerkin procedure for solving the problem (1.1) and (1.2) begins by selecting composite Sinc functions appropriate to the interval $(0, 1)$ so that their translates form a basis functions for the expansion of the approximate solution $y(x)$. A through review of properties of the Sinc function and the general

Sinc–Galerkin method can be found in [12,15,20,23]. The next section contains an overview of properties of the Sinc function that are used in the sequel.

2.1. Sinc interpolation

The goal of this section is to recall notation and definitions of the Sinc function, state some known results, and derive useful formulas that are important for this paper. First denote the set of all integers, the set of all real numbers, and the set of all complex numbers by **Z**, **R** and **C**, respectively.

- $\mathrm{sinc}(z) = \sin(\pi z)/\pi z, \quad z \in \mathbf{Z}$
  Note that $|\mathrm{sinc}(x)| \leqslant 1$ for any $x \in \mathbf{R}$.
- $S(k,h)(z) = \mathrm{sinc}[(z - kh)/h]$, $z \in \mathbf{Z}$, $h > 0$
- $C(f,h) = \sum_{k=-\infty}^{\infty} f(hk)S(k,h)(x), h > 0$
  Here, $C(f,h)$ is called the Whittaker cardinal expansion of $f(x)$ whenever this series converges.
- $C_N(f,h) = \sum_{k=-N}^{N} f(kh)S(k,h)$.

The properties of Whittaker cardinal expansions have been studied and are thoroughly surveyed in [23]. These properties are derived in the infinite strip D_d of the complex plane where for $d > 0$

$$D_d = \left\{\zeta = \xi + \mathrm{i}\eta : |\eta| < d \leqslant \frac{\pi}{2}\right\}.$$

Approximations can be constructed for infinite, semi-finite, and finite intervals. To construct approximations on the interval $(0,1)$, which are used in this paper, consider the conformal map

$$\phi(x) = \ln\left(\frac{x}{1-x}\right). \tag{2.1}$$

The map ϕ carries the eye-shaped region

$$D_E = \left\{z = x + \mathrm{i}y : \left|\arg\left(\frac{z}{1-z}\right)\right| < d \leqslant \frac{\pi}{2}\right\},$$

onto the infinite strip D_d. The composition

$$S_j(x) = S(j,h) \circ \phi(x),$$

define the basis elements for Eq. (1.1) on the intervals $(0,1)$.

The "mesh sizes" h represents the mesh size in D_d for the uniform grids $\{kh\}$, $k = 0, \pm1, \pm2,\ldots$ The Sinc grid points $z_k \in (0,1)$ in D_E will be denoted by x_k because they are real. The inverse images of the equi-spaced grids are

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}, \tag{2.2}$$

and the inverse of ϕ is denoted by ψ.

$$\psi(w) = \frac{e^{w}}{1 + e^{w}},$$

The Sinc–Galerkin method requires the derivatives of composite sinc functions be evaluated at the nodes. We need the following two lemmas.

Lemma 2.1. (See [22]). *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (2.1). Then*

$$\delta_{jk}^{(0)} = [S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \tag{2.3}$$

$$\delta_{jk}^{(1)} = h\frac{d}{d\phi}[S(j,h) \circ \phi(x)]|_{x=x_k} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases} \tag{2.4}$$

$$\delta_{jk}^{(2)} = h^2 \frac{d^2}{d\phi^2}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & j=k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j\neq k. \end{cases} \tag{2.5}$$

$$\delta_{jk}^{(3)} = h^3 \frac{d^3}{d\phi^3}[S_j]\Big|_{x=x_k} = \begin{cases} 0, & j=k, \\ \frac{(-1)^{k-j}}{(k-j)^3}[6-\pi^2(k-j)^2], & j\neq k, \end{cases} \tag{2.6}$$

and

$$\delta_{jk}^{(4)} = h^4 \frac{d^4}{d\phi^4}[S_j]\Big|_{x=x_k} = \begin{cases} \frac{\pi^4}{5}, & j=k, \\ \frac{-4(-1)^{k-j}}{(k-j)^4}[6-\pi^2(k-j)^2], & j\neq k. \end{cases} \quad \square \tag{2.7}$$

With some computations, one can prove the following lemma.

Lemma 2.2. [13]. *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_d, given by (2.1). Then*

$$\delta_{jk}^{(5)} = h^5 \frac{d^5}{d\phi^5}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} 0, & j=k, \\ \kappa_{jk}, & j\neq k, \end{cases} \tag{2.8}$$

where

$$\kappa_{jk} = \frac{(-1)^{k-j}}{(k-j)^5}[120-20\pi^2(k-j)^2+\pi^4(k-j)^4],$$

$$\delta_{jk}^{(6)} = h^6 \frac{d^6}{d\phi^6}[S(j,h)\circ\phi(x)]|_{x=x_k} = \begin{cases} \frac{\pi^6}{7}, & j=k, \\ \mu_{jk}, & j\neq k, \end{cases} \tag{2.9}$$

where

$$\mu_{jk} = \frac{-6(-1)^{k-j}}{(k-j)^6}[120-20\pi^2(k-j)^2+\pi^4(k-j)^4]. \quad \square$$

In Eqs. (2.3)–(2.9) h is step size and x_k is a sinc grid point as in (2.2).

2.2. Linear boundary value problems

Consider a linear, BVP of the form

$$y^{(2n)}(x) + \sigma(x)y(x) = f(x) \tag{2.10}$$

subject to boundary conditions

$$y^{(i)}(0) = 0, \quad y^{(i)}(1) = 0, \quad i = 0,1,\ldots,n-1. \tag{2.11}$$

Assume an approximate solution of the form

$$y_m(x) = \sum_{j=-M}^{N} c_j S_j(x), \quad m = M+N+1. \tag{2.12}$$

The unknown coefficients $\{c_j\}_{-M}^{N}$ in (2.12) are determined by orthogonalizing the residual $Ly_m - f$ with respect to the functions $\{S_k\}_{k=-M}^{N}$. This yields the discrete system

$$\langle Ly_m - f, S_k\rangle = 0, \tag{2.13}$$

for $k = -M, -M+1,\ldots,N$. The weighted inner product $\langle,\rangle$ is taken to be

$$\langle \eta,\xi\rangle = \int_0^1 \eta(x)\xi(x)w(x)\mathrm{d}x. \tag{2.14}$$

Here $w(x)$ plays the role of a weight function which is chosen depending on the boundary conditions, the domain, and the differential equation. For the case of $2n$ order boundary value problems, it is convenient to take

$$w(x) = \frac{1}{(\phi'(x))^n}.$$

A complete discussion on the choice of the weight function can be found in [14].

The most direct development of the discrete system for Eq. (2.10) is obtained by substituting (2.12) into (2.10). The system can then be expressed in integral form via (2.14). This approach, however, obscures the analysis which is necessary for applying Sinc quadrature formulas to (2.13). An alternative approach is to analyze instead

$$\langle y^{(2n)}, S_k\rangle + \langle \sigma(x)y, S_k\rangle = \langle f, S_k\rangle, \quad k = -M, \ldots, N. \tag{2.15}$$

The method of approximating the integrals in (2.15) begins by integrating by parts to transfer all derivatives from u to S_k. The approximation of the second and third inner of (2.15)

$$\langle G, S_k\rangle = h\frac{G(x_k)w(x_k)}{\phi'(x_k)}, \tag{2.16}$$

where G is $\sigma(x)y$ or f.

We need the following two theorem.

Theorem 2.1. [13]. *The following relations hold*

$$\langle y^{(2n)}, S_k\rangle = h\sum_{j=-M}^{N}\sum_{i=0}^{2n}\frac{y(x_j)}{\phi'(x_j)h^i}\delta_{kj}^{(i)}g_{2n,i}(x_j). \quad \square \tag{2.17}$$

Replacing each term of (2.15) with the approximation defined in (2.17) and (2.16) respectively, and replacing $y(x_j)$ by c_j, and dividing by h, we obtain the following theorem.

Theorem 2.2. *If the assumed approximate solution of the boundary-value problem (2.10), (2.11) is (2.12), then the discrete Sinc–Galerkin system for the determination of the unknown coefficients $\{c_j\}_{j=-M}^{N}$ is given by*

$$\sum_{j=-M}^{N}\sum_{i=0}^{2n}\frac{1}{h^i}\delta_{kj}^{(i)}\frac{g_{2n,i}(x_j)}{\phi'(x_j)}c_j + \frac{\sigma(x_k)w(x_k)}{\phi'(x_k)}c_k = \frac{f(x_k)w(x_k)}{\phi'(x_k)} \quad k = -M, \ldots, N. \tag{2.18}$$

The following notation will be necessary for writing down the system. Let $\mathbf{D}(g)$ be the $m \times m$ diagonal matrix

$$\mathbf{D}(g) = \begin{pmatrix} g(x_{-M}) & & & & & \\ & g(x_{-M+1}) & & & & \\ & & \cdots & & & \\ & & & g(x_0) & & \\ & & & & \cdots & \\ & & & & & g(x_N) \end{pmatrix}$$

and define the $m \times m$ matrices $\mathbf{I}^{(p)}$ (see [19]) for $0 \leqslant p \leqslant 2n$ by

$$I^{(p)} = [\delta_{jk}^{(p)}], \quad j, k = -M, \ldots, N.$$

Let $\mathbf{c}$ be the m-vector with jth component given by c_j, and $\mathbf{1}$ is an m-vector each of whose components is 1. In this notation the system in (2.18) takes the matrix form

$$\mathbf{A}\mathbf{c} = \Theta, \tag{2.19}$$

where

$$\Theta = \mathbf{D}\left(\frac{wf}{\phi'}\right)\mathbf{1} \tag{2.20}$$

and

$$\mathbf{A} = \sum_{i=0}^{2n} \frac{1}{h^i} \mathbf{I}^{(i)} \mathbf{D}(a_i),$$

the functions $a_i(x)$, are given by

$$a_0 = \frac{g_{2n,0} + \sigma w}{\phi'}$$

and

$$a_i = \frac{g_{2n,i}}{\phi'}, \quad 1 \leqslant i \leqslant 2n.$$

Now we have a linear system of $m = M + N + 1$ equation of the m unknown coefficients, namely, $\{c_j\}_{j=-M}^{N}$. We can obtain the coefficient of the approximate solution by solving this linear system by Q-R method. The solution $\mathbf{c} = (c_{-M}, \ldots, c_N)^{\mathrm{T}}$ gives the coefficients in the approximate Sinc–Galerkin solution $u_m(x)$ of $u(x)$.

2.3. Nonlinear boundary value problems

Consider a nonlinear, BVP of the form

$$y^{(2n)}(x) + \sigma(x)(x) y^{v}(x) = f(x). \tag{2.21}$$

The approximate solution for $u(x)$ is represented by the formula

$$y_m(x) = \sum_{j=-M}^{N} c_j S_j(x), \quad m = M + N + 1. \tag{2.22}$$

The unknown coefficients c_j in Eq. (2.22) are determined by orthogonalizing the residual with respect to the basis functions, i.e.

$$\langle y^{(2n)}, S_k \rangle + \langle \sigma(x) y^{v}, S_k \rangle = \langle f, S_k \rangle, \quad -M \leqslant k \leqslant N. \tag{2.23}$$

We need the following lemma:

Lemma 2.3. [14]. *The following relations hold*

$$\langle \sigma(x) y^{v}, S_k \rangle = h \frac{w(x_k) y^{v}(x_k) \sigma(x_k)}{\phi'(x_k)}. \quad \square \tag{2.24}$$

Replacing each term of (2.23) with the approximations defined in (2.16), (2.17) and (2.24) and replacing $y(x_j)$ by c_j and dividing by h, we obtain the following theorem.

Theorem 2.3. If the assumed approximate solution of the boundary-value problem (2.21) and (2.11) is (2.22), then the discrete Sinc–Galerkin system for the determination of the unknown coefficients $\{c_j\}_{j=-M}^{N}$ is given by

$$\sum_{j=-M}^{N} \sum_{i=0}^{2n} \frac{1}{h^i} \delta_{kj}^{(i)} \frac{g_{2n,i}(x_j)}{\phi'(x_j)} c_j + \frac{\sigma(x_k) w(x_k)}{\phi'(x_k)} c_k^{v} = \frac{f(x_k) w(x_k)}{\phi'(x_k)} \quad k = -M, \ldots, N. \tag{2.25}$$

Using the notation in the pervious section. Let $\mathbf{c}^{v}$ be the m-vectors with jth component given by c_j^{v} and $\mathbf{1}$ is an m-vector each of whose components is 1. The system in (2.25) takes the matrix form

$$\mathbf{A}\mathbf{c} + \mathbf{E}\mathbf{c}^{v} = \Theta,$$

where

$$\mathbf{E} = \mathbf{D}\left(\frac{\sigma w}{\phi'}\right)$$

and

$$\mathbf{A} = \sum_{j=0}^{2n} \frac{1}{h^j} \mathbf{I}^{(j)} \mathbf{D}\left(\frac{g_{2n,j}}{\phi'}\right)$$

and Θ are defined by Eq. (2.20). Now we have a nonlinear system of $m = M + N + 1$ equation of the m unknown coefficients, namely, $\{c_j\}_{j=-M}^{N}$. We can obtain the coefficient of the approximate solution by solving this nonlinear system by *Newton's method* [14]. The solution $\mathbf{c} = (c_{-M}, \ldots, c_N)^{\mathrm{T}}$ gives the coefficients in the approximate Sinc–Galerkin solution $u_m(x)$ of $u(x)$.

2.4. Treatment of the boundary conditions

In the previous section the development of the Sinc–Galerkin technique for homogeneous boundary conditions provided a practical approach since the sinc function composed with various conformal mappings, $S(j,h) \circ \phi$, are zero at the endpoints of the interval. If the boundary conditions are nonhomogeneous, then these conditions need be converted to homogeneous ones via an interpolation by a known function (see [13]).

3. The modified decomposition method

The decomposition method has been shown [1,2] to solve effectively, easily, and accurately a large class of linear and nonlinear, ordinary or partial, deterministic or stochastic differential equations with approximate which converge rapidly to accurate solutions. The method is well-suited to physical problems since it makes unnecessary the linearization, perturbation, and other restrictive methods and assumptions which may change the problem being solved, sometimes seriously.

3.1. Analysis of the method

In an operator form, Eq. (1.1) can be written as

$$Ly = f(x, y),$$

where the differential operator L is given by

$$L = \frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}. \tag{3.1}$$

The inverse operator L^{-1} is therefore considered a $2n$-fold integral operator defined by

$$L^{-1}(\cdot) = \underbrace{\int_0^x}_{(2n)\text{times}} (\cdot) \underbrace{\mathrm{d}x}_{(2n)\text{times}}.$$

Operating with L^{-1} on (3.1) and using the boundary conditions at $x = 0$ yields

$$y(x) = \sum_{j=0}^{2n-1} \frac{\alpha_j}{j!} x^j + L^{-1}[f(x, y)], \tag{3.2}$$

where

$$\alpha_j = y^{(j)}(0), \quad j = n, n+1, \ldots, 2n-1$$

are constants that will be determined later by using the boundary conditions at $x = 1$. The other constants $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are prescribed in (1.2).

The Adomian decomposition method defines the solution $y(x)$ of (1.1) by the decomposition series

$$y(x) = \sum_{m=0}^{\infty} y_m(x) \tag{3.3}$$

and the nonlinear function $f(x,y)$ by an infinite series of polynomials

$$f(x,y) = \sum_{m=0}^{\infty} A_m, \tag{3.4}$$

where the components $y_m(x)$ of the solution $y(x)$ will be determined recurrently, and A_m are the so-called Adomian polynomials that can be constructed for various classes of nonlinearity according to specific algorithms set by Adomain [1,2]. Recently, a new algorithm for calculating these polynomials was derived in [28]. Substituting (3.3) and (3.4) into (3.2) yields

$$\sum_{m=0}^{\infty} y_m(x) = \sum_{j=0}^{2n-1} \frac{\alpha_j}{j!} x^j + L^{-1}\left(\sum_{m=0}^{\infty} A_m\right). \tag{3.5}$$

The decomposition method identifies the zeroth component $y_0(x)$ by all terms that arise from the boundary condition at $x = 0$ and from integrating the source term if exists. Based on this identification, the method formally admits the use of the recursive relation

$$\begin{aligned} y_0(x) &= \sum_{j=0}^{2n-1} \frac{A_j}{j!} x^j, \\ y_{k+1}(x) &= L^{-1}(A_k), \quad k \geqslant 0, \end{aligned} \tag{3.6}$$

for the determination of the components $y_m(x)$ of $y(x)$.

$$\begin{aligned} y_0(x) &= \alpha_0, \\ y_1(x) &= \sum_{j=1}^{2n-1} \frac{\alpha_j}{j!} x^j + L^{-1}(A_0), \\ y_{k+1}(x) &= L^{-1}(A_k), \quad k \geqslant 1. \end{aligned} \tag{3.7}$$

Having determined the components $y_m(x)$, $m \geqslant 0$ recurrently, the series solution of $y(x)$ follows immediately with the constants $\alpha_j, j = n, n+1, \ldots, 2n-1$ are as yet undetermined.

An important point to be made here is that we can elegantly determine the components $y_m(x)$ as far as we like to enhance the accuracy of the approximation. The approximate

$$\phi_n = \sum_{k=0}^{n-1} y_k$$

can be used to approximate the solution.

Our aim is now to determine the constants $\alpha_j, j = n, n+1, \ldots, 2n-1$. Imposing the remaining the boundary conditions at $x = 1$ on the approximate ϕ_n leads to an algebraic system of equations. This system needs only be solved to obtain approximations to the constants $\alpha_j, j = n, n+1, \ldots, 2n-1$. Having determined these constants, the numerical solution of the $2n$-order boundary value problem follows immediately upon substituting the resulting components in (3.3).

4. Numerical examples

The examples reported in this section were selected from a large collection of problems to which the Sinc–Galerkin and modified decomposition methods could be applied. For purposes of comparison, contrast and performance, examples with known solutions were chosen.

We give four examples, two linear and two nonlinear. These examples demonstrate how the Sinc–Galerkin method outperforms the modified decomposition method.

For the Sinc–Galerkin, d is taken to be $\pi/2$. The step size h and the summation limits M, and N are selected so that the error in each coordinate direction is a asymptotically balanced. Once M is chosen, the step size and remaining summation limit can be determined as follows:

$$h=\sqrt{\frac{\pi d}{\alpha M}}\quad\text{and}\quad N=\left[\left|\frac{\alpha M}{\beta}\right|\right].$$

We use the absolute error which is defined as

$$E_S=|u_{\text{exact solution}}-U_{\text{Sinc–Galerkin}}|$$

and

$$E_d=|u_{\text{exact solution}}-U_{\text{modified decomposition}}|$$

Example 4.1. Consider the Linear BVP

$$y^{(4)}=-2e^x+3y,\quad 0<x<1 \tag{4.1}$$

subject to the boundary conditions

$$\begin{aligned}&y(0)=1,\quad y(1)=e,\\&y'(0)=1,\quad y'(1)=e.\end{aligned}$$

The exact solution for this problem is

$$y=e^x.$$

For modified decomposition method, Eq. (4.1) may be written in an operator form by

$$Ly=-2e^x+3y,\quad 0<x<1. \tag{4.2}$$

Operating with L^{-1} on (4.2) and using the boundary conditions at $x=0$, yields

$$y(x)=1+x+\frac{\alpha_2}{2}x^2+\frac{\alpha_3}{3!}x^3-2L^{-1}e^x+3L^{-1}[y(x)], \tag{4.3}$$

where the inverse operator L^{-1} is a four-fold integral operator and

$$\alpha_2=y''(0),\quad\text{and}\quad\alpha_3=y^{(3)}(0)$$

are constants that will be determined later by using the boundary conditions at $x=1$. Substituting the decomposition series (3.3) for $y(x)$ into (4.3) gives

$$\sum_{m=0}^{\infty}y_m(x)=1+x+\frac{\alpha_2}{2}x^2+\frac{\alpha_3}{3!}x^3-2L^{-1}(e^x)+3L^{-1}\left(\sum_{m=0}^{\infty}y_m\right). \tag{4.4}$$

To determine the components $y_m(x), m\geqslant 0$, the modified decomposition method introduces the recursive relation

$$\begin{aligned}&y_0(x)=1,\\&y_1(x)=x+\frac{\alpha_2}{2}x^2+\frac{\alpha_3}{3!}x^3-2L^{-1}e^x+3L^{-1}[y_0(x)],\\&y_{k+1}(x)=3L^{-1}[y_k(x)],\quad k\geqslant 1.\end{aligned} \tag{4.5}$$

This gives

$$\begin{aligned}&y_0(x)=1,\\&y_1(x)=x+\frac{\alpha_2}{2}x^2+\frac{\alpha_3}{3!}\alpha_3x^3+\frac{1}{4!}x^4-2\left[\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\frac{x^8}{8!}+\frac{x^9}{9!}+\frac{x^{10}}{10!}+\cdots\right],\\&y_2(x)=3L^{-1}[y_1]\\&\qquad=3\left[\frac{x^5}{5!}+\alpha_2\frac{x^6}{6!}+\alpha_3\frac{x^7}{7!}+\frac{x^8}{8!}-2\left(\frac{x^9}{9!}+\frac{x^{10}}{10!}+\frac{x^{11}}{11!}+\frac{x^{12}}{12!}+\cdots\right)\right],\end{aligned}$$

$$y_3(x) = 3L^{-1}[y_2] \\ = 9\left[\frac{x^9}{9!} + \alpha_2 \frac{x^{10}}{10!} + \alpha_3 \frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \cdots\right]. \tag{4.6}$$

Consequently, the approximation of $y(x)$ is given by

$$y(x) = 1 + x + \frac{\alpha_2}{2}x^2 + \frac{\alpha_3}{3!}x^3 + \frac{1}{4!}x^4 + \frac{x^5}{5!} + (3\alpha_2 - 2)\frac{x^6}{6!} + (3\alpha_3 - 2)\frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + (9\alpha_2 - 8)\frac{x^{10}}{10!} \\ + (9\alpha_3 - 8)\frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \cdots \tag{4.7}$$

It remains to determine approximations to the constants α_2, and α_3. This can be achieved by imposing the boundary conditions at $x = 1$ on the 4-term approximate ϕ_4 derived from (4.7). According, we obtain the algebraic system, by solving this algebraic system gives

$$\alpha_2 = 0.99999999461435 \quad \text{and} \quad \alpha_3 = 1.00000001452276.$$

The series solution is

$$y(x) = 1 + x + 0.49999999730717x^2 + 0.16666666908713x^3 + \frac{x^4}{4!} + \frac{x^5}{5!} + 0.00138888886645x^6 \\ + 1.984127070571985 \times 10^{-4}x^7 + \frac{x^8}{8!} + \frac{x^9}{9!} + 2.755731788825919 \times 10^{-7}x^{10} \\ + 2.505211165987355 \times 10^{-8}x^{11} + \cdots \tag{4.8}$$

For Sinc–Galerkin method, the parameters selected $\alpha = \beta = \frac{3}{2}$, $M = N = 100$. Table 1 exhibits the exact, the Sinc–Galerkin and the modified decomposition solutions.

Maximum absolute error are tabulated in Table 2 for Sinc–Galerkin together with the modified decomposition method.

Example 4.2. Consider the linear BVP

$$y^{(6)} - y = -6e^x, \quad 0 < x < 1 \tag{4.9}$$

subject to the boundary conditions

Table 1
The exact, Sinc–Galerkin and modified decomposition solutions for example 1

x	Exact solution	Sinc–Galerkin	Modified decomposition
0.0	1.0	1.0	1.0
0.0568	1.053375741	1.053375742	1.053375742
0.1084	1.150388830	1.150388830	1.150388831
0.1970	1.385553912	1.385553911	1.385553917
0.3313	1.507119172	1.507119170	1.507119178
0.5	1.648721262	1.648721259	1.648721270
0.6687	1.961677536	1.961677532	1.961677549
0.8030	2.411381912	2.411381909	2.411381933
0.9432	2.601790833	2.601790832	2.601790857
0.9927	2.697970947	2.697970947	2.697970973
1.0	2.718281828	2.718281828	2.718281825

Table 2
Maximum absolute error for example 1

Sinc–Galerkin method E_S	The decomposition method E_d
0.37E−008	0.25E−007

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1,$$
$$y(1) = 0, \quad y'(1) = -e, \quad y''(1) = -2e.$$

The exact solution for this problem is

$$y = (1 - x)e^x.$$

For the modified decomposition method, the series solution is

$$\begin{aligned} y(x) = 1 - \frac{x^2}{2!} &- 0.33333405442309x^3 - 0.12499838066502x^4 - 0.03333425662668x^5 \\ &- 0.00694444444444x^6 - 0.00138888888889x^7 - 1.984128057177215 \times 10^{-4}x^8 \\ &- 2.480148020271310 \times 10^{-5}x^9 - 2.755762454585902 \times 10^{-6}x^{10} + \cdots \end{aligned} \tag{4.10}$$

For Sinc–Galerkin method, the parameters selected $\alpha = \beta = \frac{5}{2}$, $M = N = 100$. Table 3 exhibits the exact, the Sinc–Galerkin and the modified decomposition solutions. Maximum absolute error are tabulated in Table 4 for Sinc–Galerkin together with the modified decomposition method.

Example 4.3. [26] Consider the nonlinear BVP

$$y^{(6)} = e^x y^2, \quad 0 < x < 1 \tag{4.11}$$

subject to the boundary conditions

$$y(0) = 1, \quad y'(0) = -1, \quad y''(0) = 1,$$
$$y(1) = e^{-1}, \quad y'(1) = -e^{-1}, \quad y''(1) = e^{-1}.$$

The exact solution for this problem is

$$y = e^{-x}.$$

Eq. (2.6) may be written in an operator form by

$$Ly = e^x y^2, \quad 0 < x < 1. \tag{4.12}$$

Operating with L^{-1} on (4.12) and using the boundary conditions at $x = 0$, yields

Table 3
The exact, Sinc–Galerkin and modified decomposition solutions for example 2

x	Exact solution	Sinc–Galerkin solution	Modified decomposition solution
0.0	1.0	1.0	1.0
0.0567	0.9983304	0.9983312	0.9983304
0.1970	0.9778484	0.9778392	0.9778484
0.3961	0.8974062	0.8973994	0.8974058
0.4302	0.8761045	0.8761012	0.8761039
0.5	0.8243606	0.8243606	0.8243589
0.6038	0.7246719	0.7246710	0.7246657
0.8772	0.2952306	0.2952287	0.2951418
0.9502	0.1287940	0.1287948	0.1286370
0.9927	0.0196991	0.0196991	0.0194846
1.0	0.00	0.00	0.0

Table 4
Maximum absolute error for example 2

Sinc–Galerkin method E_S	The decomposition method E_d
0.92E−005	0.21E−003

$$y(x) = 1 - x + \frac{1}{2}x^2 + \frac{1}{3!}\alpha_3 x^3 + \frac{1}{4!}\alpha_4 x^4 + \frac{1}{5!}\alpha_5 x^5 + L^{-1}[e^x y^2(x)], \tag{4.13}$$

where

$$\alpha_3 = y'''(0), \quad \alpha_4 = y^{(4)}(0), \quad \text{and} \quad \alpha_5 = y^{(5)}(0)$$

are constants that will be determined later by using the boundary conditions at $x = 1$. Substituting the decomposition series (3.3) for $y(x)$ into (4.13) gives

$$\sum_{m=0}^{\infty} y_m(x) = 1 - x + \frac{1}{2}x^2 + \frac{1}{3!}\alpha_3 x^3 + \frac{1}{4!}\alpha_4 x^4 + \frac{1}{5!}\alpha_5 x^5 + L^{-1}\left(e^x \sum_{m=0}^{\infty} A_m\right), \tag{4.14}$$

where A_m are the so-called Adomian polynomials that represent the nonlinear term $y^2(x)$.

To determine the components $y_m(x)$, $m \geqslant 0$, the modified decomposition method introduces the recursive relation

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= -x + \frac{1}{2}x^2 + \frac{1}{3!}\alpha_3 x^3 + \frac{1}{4!}\alpha_4 x^4 + \frac{1}{5!}\alpha_5 x^5 + L^{-1}[e^x A_0(x)], \\ y_{k+1}(x) &= L^{-1}[e^x A_k(x)], \quad k \geqslant 1. \end{aligned} \tag{4.15}$$

It is useful to the first few Adomian polynomials A_m for the nonlinear operator $F(y) = y^2$. Following the analysis of [1,27] yields:

$$\begin{aligned} A_0 &= F(y_0) = y_0^2(x), \\ A_1 &= y_1(x)F'(y_0) = 2y_0(x)y_1(x), \\ A_2 &= y_2 F'(y_0) + \frac{y_1^2}{2!}F''(y_0) = 2y_0(x)y_2(x) + y_2^2(x) \end{aligned} \tag{4.16}$$

and so on for other polynomials. Inserting (4.16) into (4.15) yields

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= -x + \frac{1}{2}x^2 + \frac{1}{3!}\alpha_3 x^3 + \frac{1}{4!}\alpha_4 x^4 + \frac{1}{5!}\alpha_5 x^5 + L^{-1}[e^x A_0(x)], \\ &= -x + \frac{1}{2}x^2 + \frac{1}{3!}\alpha_3 x^3 + \frac{1}{4!}\alpha_4 x^4 + \frac{1}{5!}\alpha_5 x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \frac{1}{9!}x^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}x^{11} + \cdots \\ y_2(x) &= L^{-1}[e^x A_1] \\ &= -\frac{x^7}{2520} - \frac{x^8}{20160} + \frac{\alpha_3 x^9}{181440} + \left(\frac{\alpha_3}{453600} + \frac{\alpha_4}{1814400} + \frac{1}{907200}\right)x^{10} \\ &\quad + \left(\frac{\alpha_3}{1995840} + \frac{\alpha_4}{3991680} + \frac{\alpha_5}{19958400} + \frac{1}{3991680}\right)x^{11} + \cdots \end{aligned} \tag{4.17}$$

Consequently, the approximation of $y(x)$ is given by

$$\begin{aligned} y(x) &= 1 - x + \frac{1}{2}x^2 + \frac{1}{3!}\alpha_3 x^3 + \frac{1}{4!}\alpha_4 x^4 + \frac{1}{5!}\alpha_5 x^5 + \frac{1}{6!}x^6 + \left(\frac{1}{7!} - \frac{1}{2520}\right)x^7 + \left(\frac{1}{8!} - \frac{1}{20160}\right)x^8 \\ &\quad + \left(\frac{1}{9!} + \frac{\alpha_3}{181440}\right)x^9 + \left(\frac{1}{10!} + \frac{\alpha_3}{453600} + \frac{\alpha_4}{1814400} + \frac{1}{907200}\right)x^{10} \\ &\quad + \left(\frac{1}{11!} + \frac{\alpha_3}{1995840} + \frac{\alpha_4}{3991680} + \frac{\alpha_5}{19958400} + \frac{1}{3991680}\right)x^{11} + \cdots \end{aligned} \tag{4.18}$$

It remains to determine approximations to the constants α_3, α_4 and α_5. This can be achieved by imposing the boundary conditions at $x = 1$ on the 3-term approximate ϕ_3 derived from (4.18). According, we obtain the algebraic system, by solving this algebraic system gives

$\alpha_3 = -0.99816409, \quad \alpha_4 = 0.98167470, \quad \text{and} \quad \alpha_5 = -0.939073710.$

The series solution is

$$y(x) = 1 - x + \frac{1}{2}x^2 - 0.1663606817x^3 + 0.0409031125x^4 - 0.007825609x^5 + \frac{1}{720}x^6 - \frac{1}{5040}x^7 - \frac{1}{40320}x^8 - 0.000002745x^9 - 0.0000002816x^{10} - 0.00000002567x^{11} + \cdots \quad (4.19)$$

For Sinc–Galerkin method, the parameters selected $\alpha = \beta = \frac{3}{2}$, $M = N = 100$. Table 5 exhibits the exact, the Sinc–Galerkin and the modified decomposition solutions.

Maximum absolute error are tabulated in Table 6 for Sinc–Galerkin together with the modified decomposition method.

Example 4.4. [5,9,14,25]. Consider the nonlinear BVP

$$y^{(4)} = 6e^{-4y} - \frac{12}{(1+x)^4}, \quad 0 < x < 1$$

subject to boundary conditions

$$y(0) = 0, \quad y(1) = \ln 2,$$
$$y'(0) = 1, \quad y'(1) = 0.5,$$

which has the exact solution given by

$$y(x) = \ln(1 + x).$$

For Sinc–Galerkin method, the parameters selected $\alpha = \beta = \frac{1}{2}$, $M = N = 100$. Maximum absolute error are tabulated in Table 7 for Sinc–Galerkin together with the modified decomposition method.

Table 5
The exact, Sinc–Galerkin and modified decomposition solutions for example 3

x	Exact solution	Sinc–Galerkin	Modified decomposition method
0.0	1.0	1.0	1.0
0.0568	0.93174137	0.93174138	0.93174146
0.1084	0.78922802	0.78922802	0.78923005
0.1970	0.71806154	0.71806156	0.71806548
0.3313	0.64294248	0.64294241	0.64294825
0.5	0.60653065	0.60653060	0.60653685
0.6687	0.54069496	0.54069494	0.54070055
0.8030	0.48753168	0.48753163	0.48753536
0.9432	0.44802779	0.44802777	0.44802964
0.9927	0.36894783	0.36894781	0.36894789
1.0	0.36787944	0.36787939	0.36787949

Table 6
Maximum absolute error for example 3

Sinc–Galerkin method E_S	The decomposition method E_d
0.51E−007	0.57E−005

Table 7
Maximum absolute error for example 4

Sinc–Galerkin method E_S	The decomposition method E_d
0.51E−013	0.57E−008

5. Conclusions

All computations associated with the above examples were performed by using **MATLAB**. Although the modified decomposition method has been shown to be a powerful numerical tool for fast and accurate solutions of differential equations, in many instances the Sinc–Galerkin method seems to give better results even nonlinear; see also [14].

Although the Sinc–Galerkin solution required slightly more computational effort than the modified decomposition solution, it resulted in more accurate results, especially in the presence of singularities [13]. This may be attributed to the fact that in the Sinc method in order to obtain an error of order $O(e^{-(kN)^{1/2}})$ for some k, the solution and the nonhomogeneous term have to be nice and smooth.

References

[1] G. Adomain, Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, 1994.
[2] G. Adomain, A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. 135 (1988) 501–544.
[3] G. Adomain, R. Rach, Analytic solution of nonlinear boundary value problems in several dimensions by decomposition, J. Math. Anal. Appl. 174 (1993) 118–137.
[5] R. Agarwal, Boundary Value Problems for High Ordinary Differential Equations, World Scientific, Singapore, 1986.
[6] N. Bellomo, R. Monaco, A comparison between Adomian's decomposition methods and perturbation techniques for nonlinear random differential equations, J. Math. Anal. Appl. 110 (1985) 495502.
[7] B. Bialecki, Sinc-Collocation methods for two-point boundary value problems, IMA J. Numer. Anal. 11 (1991) 357–375.
[8] A. Boutayeb, E. Twizell, Finite-difference methods for the solution of special eighth-order boundary value problems, Intern. J. Comput. Math. 48 (1993) 63–75.
[9] M. Chawal, C. Katti, Finite difference methods for two-point boundary value problems involving high order differential equations, BIT 19 (1979) 27–33.
[10] J.D. Dockery, Numerical solution of travelling waves for reaction-Diffusion equations via the Sinc–Galerkin method, in: K. Bowers, J. Lund (Eds.), Computation and Control: Proceedings of the Bozeman Conference, Bozeman, Montana, 1993, Noston, Birkhauser, 1993, pp. 95–113.
[11] J.Y. Edwards, J.A. Roberts, N.J. Ford, A comparison of Adomian's decomposition method and RungeKutta methods for approximate solution of some predator prey model equations, Technical Report No. 309, Manchester Center of Computational Mathematics,1997, p. 117.
[12] M. El-Gamel, A. Zayed, A comparison between the wavelet-Galerkin and the Sinc–Galerkin methods in solving nonhomogeneous heat equations, in: Contemporary Mathematics of the American Mathematical Society, Zuhair Nashed, Otmar Scherzer (Eds.), Series, Inverse Problem, Image Analysis, and Medical Imaging, vol. 313, AMS, Providence, 2002.
[13] M. El-Gamel, J.R. Cannon, A. Zayed, Sinc–Galerkin method for solving linear Sixth order boundary-value problems, Math. Comp. 73 (2004) 1325–1343.
[14] M. El-Gamel, A. Zayed, Sinc–Galerkin method for solving nonlinear boundary-value problems, Comput. Math. Appl. 48 (2004) 1285–1298.
[15] M. El-Gamel, J. Cannon, On the solution of second order singularly-perturbed boundary value problem by Sinc–Galerkin method, Z. Angew. Math. Phys. 56 (2005) 45–58.
[16] M. El-Gamel, A numerical scheme for solving nonhomogeneuous time-dependent problems, Z. Angew. Math. Phys. 57 (2006) 369–383.
[17] M. El-Gamel, The Sinc–Galerkin method for solving singularly-perturbed reaction–diffusion problem, ETNA 23 (2006) 129–140.
[18] S. El-Sayed, M. Abdel-Aziz, A comparison of Adomian's decomposition method and wavelet-Galerkin method for solving integro-differential equations, Appl. Math. Comput. 136 (2003) 151–159.
[19] V. Grenander, G. Szego, Toeplitz Forms and Their Applications, second ed., Chelsea Publishing Co, Orlando, 1985.
[20] J. Lund, K. Bowers, Sinc Methods for Quadrature and Differential Equations, SIAM, Philadelphia, PA, 1992.
[21] A. Mohsen, M. El-Gamel, A Sinc-Collocation method for the linear Fredholm integro-differential equations, Z. Angew. Math. Phys., in press.
[22] R. Smith, G. Bogar, K. Bowers, J. Lund, The Sinc–Galerkin method for fourth-order differential equations, SIAM J. Numer. Anal. 28 (1991) 760–788.
[23] F. Stenger, Numerical Methods Based on Sinc and Analytic Functions, Springer, New York, 1993.
[24] E. Twizell, A. Boutayeb, Numerical methods for solution of sixth-order boundary value problems with applications to Benard layer eigenvalue problems, Proc. R. Soc. Lond. Ser. A 431 (1990) 433–450.
[25] E. Twizell, S. Tirmizi, Multiderivative methods for nonlinear beam problems, Commun. Appl. Numer. Meth. 4 (1988) 43–50.
[26] A. Wazwaz, The numerical solution of sixth order boundary value problems by the modified decomposition method, Appl. Math. Comput. 118 (2001) 311–325.
[27] A. Wazwaz, The modified decomposition method and Pade' approximants for solving the Thomas–Fermi equation, Appl. Math. Comput. 105 (1999) 11–19.

[28] A. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math. Comput. 111 (2000) 33–51.
[29] A. Wazwaz, A comparison between Adomian decomposition method and Taylor series method in the series solutions, Appl. Math. Comput. 79 (1998) 37–44.
[30] E. Yee, Application of the decomposition method to the solution of the reaction convection- diffusion equation, Appl. Math. Comput. 56 (1993) 127.
[31] G. Yin, Sinc-Collocation method with orthogonalization for singular problem-like poisson, Math. Comput. 62 (1994) 21–40.